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Semiclassical analysis of spectral correlations in regular billiards with point scatterers

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A semiclassical analysis is proposed to elucidate quantitatively the deviations from the predictions of the random matrix theory of the observed conditional number density in rectangular billiards with point scatterers [R. L. Weaver and D. Sornette, Phys. Rev. E **52**, 3341 (1995)]. Using the scattering cross section of the point scatterer, the spectral form factor is shown to be built on two categories of periodic orbits depending whether they are scattered or not. Our quantitative predictions are successfully compared to the observed spectral correlations in various cases of a rectangular billiard with one or several point scatterers. [S1063-651X(97)14106-X]

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It has been conjectured that the eigenvalue statistics of generic systems which are classically chaotic are identical to those of random matrices belonging to the Gaussian orthogonal ensemble (GOE). It is also accepted that regular integrable systems should display Poisson statistics. However, many classically integrable systems have been shown to exhibit spectral rigidity which is typical of GOE-like systems [1,2]. A particular example is the singular quantum billiard introduced by Seba [2]. It consists of a rectangular billiard with an isotropic point scatterer. In previous works [3,4], conditions for the appearance of level repulsion or spectral rigidity have been discussed, and a quantitative prediction for the range of GOE-like statistics was proposed in Ref. [3], relying upon a proper definition of the scattering cross section of the scatterer. Here we present a semiclassical analysis of the spectral form factor in order to provide quantitative predictions for the conditional density of levels in rectangular billiards with one or several point scatterers. Those predictions are compared with the statistics of eigenvalues numerically evaluated through the method introduced in Ref. [3].

A regular billiard with a point scatterer remains fully integrable. At finite time, and except for a set of measure zero, the infinitesimal scatterer does not affect ray paths. One might therefore presume that the statistics remain Poissonian. This may be contrasted with the usual asymptotic consideration of the fully chaotic Sinai billiard, in which the wavelength is taken to zero while the radius of the removed arc is kept finite, and the statistics are GOE. If, however, one investigates the *distinguished limit* in which scatterer size is taken to zero at the same rate that wavelength is taken to zero, one recovers a regime of considerable current interest. This is precisely the limit implicit in recent studies [2–4] of the isotropic point scatterer with finite cross section in a rectangular billiard. The limit could presumably also be reached by study of a Sinai billiard with small arc at large, but finite, energy. We know of no such studies.

First, we briefly recall that a point scatterer in any dimension $D \geqslant 2$ cannot be represented through a scattering potential. The scatterer is in fact properly defined by its t matrix in terms of which its cross section is readily obtained. For the Helmholtz wave equation in two dimensions, the latter is a length σ which depends on the frequency ω and on a "strength" dimensionless parameter α as [3]

$$\sigma = 4/(\omega\sqrt{1+4\alpha^2}). \tag{1}$$

This form is readily obtained by imposing flux conservation between incident and scattered waves which yields a one-parameter transition strength for the scatterer. In a finite system, this procedure enables one to find the modes of the dressed system (i.e., with the point scatterer) through the modes of the bare system (i.e., without the point scatterrer) by using the Lippman-Schwinger equation relating the dressed Green's function to the undressed Green's function via the parameter α [see Eqs. (14)–(23) of Ref. [3]]. This one-parameter family of eigenvalues constitute the spectrum of the self-adjoint extensions of the Helmholtz operator in the presence of a point scatterer [4]. The procedure presented in Ref. [3] is extended to the case of several scatterers by the analysis presented in the Appendix.

To study quantitatively the spectral correlations among the eigenvalues associated with the rectangular membrane (Dirichlet boundary condition) with a point scatterer, we follow a semiclassical analysis along the lines proposed by Argaman, Imry, and Smilansky [5]. In the following, we will use the dimensionless frequency variable $x \equiv \omega/\langle\Delta\omega\rangle$, where ω is the angular frequency and $\langle\Delta\omega\rangle$ is the mean frequency spacing between adjacent modes around ω. This amounts to considering the so-called unfolded spectrum [6]. Now, the commonly considered nearest-neighbor spacing distribution, though revealing a possible level repulsion, is not very sensitive to mid-range or long-range correlations. Instead, the conditional probability $g(s)ds$ of finding a level in the interval $[x+s,x+s+ds]$, given that there is one level at x (assuming here and in the rest that the spectrum is stationary), is a true two-point measure likely to characterize spectral rigidity. The latter notion manifests itself in the slow increase of the variance of the number of levels in a given frequency interval with the mean value of this number [typically, for GOE spectra, $\Sigma^2 \equiv \langle (N-\langle N\rangle)^2\rangle \approx (2/\pi^2)\ln(2\pi\langle N\rangle)$ for large $\langle N\rangle$ whereas, for uncorrelated Poisson spectra, $\Sigma^2=\langle N\rangle$] [6]. The conditional number density $g(s)$ is equivalently expressed as $1-\delta(s)+K(s)$ where $K(s)=\langle (d[x-(s/2)]-1)(d[x+(s/2)]-1)\rangle$ is the autocovariance of the spectral density $d(x)=\Sigma_n\delta(x-x_n)$ (the mean value of which is unity). Defining the spectral form factor as the Fourier transform $\widetilde{K}(\tau)=\int ds\ e^{i\tau s}K(s)$, one may show (see Ref. [5] and also Berry's course in Ref. [7]) that a semiclassical evaluation of the form factor is obtained as a sum over periodic orbits of the corresponding billiard, which reads (in the diagonal approximation)

$$\widetilde{K}(\tau)\approx \sum_{\text{p.o.'s}} |A_j|^2\delta(\tau-\tau_j), \tag{2}$$

where the A_j's are the amplitudes and the τ_j's are the dimensionless periods ($\langle\Delta\omega\rangle T=\tau$, with T the true period) of the periodic orbits. This sum rule yields a valid approximation of the spectral form factor for values of τ much larger than $\langle\Delta\omega\rangle T_0$ (with T_0 being the period of the shortest periodic orbit) and still much smaller than 2π. For larger values of τ, another sum rule, proposed by Berry [7], shows that $\widetilde{K}\rightarrow 1$ if $\tau \gg 2\pi$. The sum rule given by Eq. (2) has proved to give the correct universal behavior as well in genetic regular systems [where $\widetilde{K}(\tau)=1$, yielding the Poisson statistics for uncorrelated spectra, see, for instance, Ref. [8]] as in chaotic systems with or without time reversal invariance [where $\widetilde{K}(\tau)\approx\tau/\pi$, which is the small τ leading behavior of the GOE form factor $1-b_{\text{GOE}}(\tau)$; see, for instance, [7]].

The key to the following argument will be to consider that the wave problem associates a finite size of the order of the cross section to the point scatterer, and that this coarse-grained scale should be taken as the diameter of a virtual disk centered at the position of the scatterer, and fixing an effective range of action of the latter on ray trajectories in the billiard. Thus, when considering the problem of a point scatterer in a regular billiard like the rectangle, one should subdivide the periodic orbits (p.o.'s) into two categories: the first group consists of p.o.'s of the original regular billiard which do not hit the disk associated to a ''coarse-grained'' scatterer while the other group consists of ''new'' trajectories which hit this disk at least once, and which are responsible, at large enough times, for the ergodic regime of the wave problem. Here we would like to stress that our approach is concerned with an integrable system with one or a few point scatterers, and not with hyperspherical rigid scatterers placed at random in a hypercubic billiard in the limit of vanishing wavelength compared to the size of the spheres. The latter problem was recently addressed in Ref. [9], and focused principally on the transition from ballistic to diffusive regimes. If sufficiently many pointlike impurities were placed in the rectangle billiard, one could eventually envisage a diffusive dynamical regime for times intermediate between the ballistic regime and the ergodic one, as considered in Ref. [5].

In a two-dimensional billiard of area S, the rate at which a typical ray hits a disk of diameter σ is given by the expression $\Gamma=\pi\sigma/\pi S$ (see, for instance, Ref. [10]). Using the leading part of Weyl's formula for the modal density at high frequencies, the mean spacing between adjacent eigenfrequencies reads $\langle\Delta\omega\rangle\approx 2\pi/\omega S$, which enables one to define the dimensionless rate

$$\gamma=\Gamma/\langle\Delta\omega\rangle=\frac{\omega\sigma}{2\pi}=\frac{2}{\pi\sqrt{1+4\alpha^2}}. \tag{3}$$

One then proceeds to evaluate the spectral form factor by summing up the contributions associated to the two categories of p.o.'s mentioned above. In the dressed rectangle, the fraction of regular periodic trajectories (belonging to the periodic orbits of the regular undressed billiard) which have not met the scatterer at time τ may be approximated to decay like $\exp(-\gamma\tau)$, thus reducing by an identical factor the amplitude A associated with those orbits in the sum rule (2). Since the sum rule in the integrable billiard yields $\widetilde{K}(\tau)\rightarrow A^2(\tau)\rho_b(\tau)=1$ [8], where $\rho_b(\tau)$ is the density (per unit dimensionless time) of p.o.'s of the bare rectangle, one deduces the ''regular part'' of the form factor of the dressed rectangle

$$\widetilde{K}_d^{\text{reg}}(\tau)\approx e^{-2\gamma\tau}. \tag{4}$$

For the other group of p.o.'s, one can reasonably assume that the corresponding ''ergodic part'' of the form factor is obtained through the ansatz consisting of multiplying the GOE form factor by the relative fraction of trajectories which have met the coarse-grained scatterer at least once before the time τ

$$\widetilde{K}_d^{\text{erg}}(\tau)=(1-e^{-\gamma\tau})[1-b_{\text{GOE}}(\tau)]. \tag{5}$$

Indeed, according to Ref. [5], a semiclassical evaluation of the spectral form factor is obtained as (in the case of time-reversal symmetry) $\widetilde{K}(\tau)\approx(\tau/\pi)P(\tau)$, where $P(\tau)$ denotes the classical probability for periodic motion. The factor (τ/π) is in fact the so-called diagonal approximation of the GOE spectral form factor $1-b_{\text{GOE}}(\tau)$, and $P(\tau)$, in the present case, should be given by $(1-e^{-\gamma\tau})$ for times large enough compared to the time of flight t_f across the billiard. Presumably, there exist deviations from the GOE (see Ref. [11]), and the contribution from the ergodic orbits could be modified to account for them, but they do not concern us as all the levels we study are in the regime $\tilde{g}\gg 1$, where $\tilde{g}\equiv 1/(\langle\Delta\omega\rangle t_f)$ is the so-called dimensionless conductance. By summing both contributions, Eqs. (4) and (5), one obtains our approximation for the complete form factor $\widetilde{K}_d(\tau)$ for the dressed rectangle. A numerical inverse Fourier transform then leads to the corresponding conditional number density $g_d(s)$. Our result differs from that obtained by Agam and

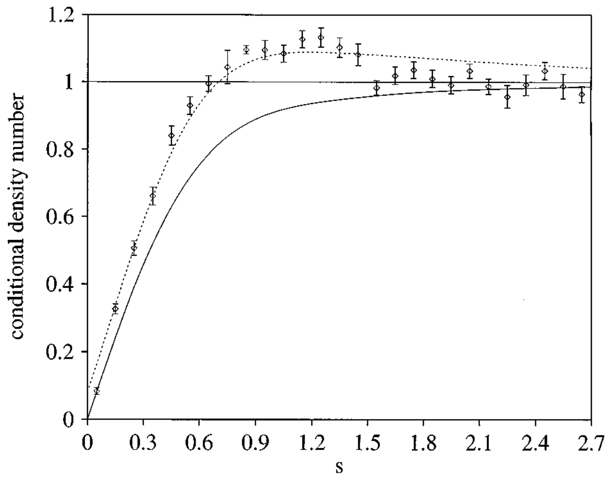

FIG. 1. Conditional number density observed among more than 10^4 even-even modes in the dressed rectangle with a single-point scatterer at the center. It is compared to an evaluation of the conditional density $g_d(s)$ (dotted line) obtained through a numerical inverse Fourier transform of our approximation for the complete form factor $\widetilde{K}_d(\tau)$ with $\gamma=2/\pi$. For the sake of comparison, the prediction of the GOE is shown (solid line).

Fishman [9] for two chief reasons: (a) our formula (4) differs by a factor of 2 from the corresponding formula (5) in the first of Ref. [9]; (b) in Ref. [9], their system has the geometry of the torus (leading to the absence of periodic orbits which are scattered only once), whereas ours has Dirichlet boundary conditions: this explains why the fourth term of formula (5.9) in the second of Ref. [9] does not appear in our result. A related result was obtained by Altland and Gefen [12] through a diagrammatic perturbative analysis of nondiffusive disordered electron systems; apart from the term accounting for the orbits that are not scattered, their result [Eq. (32) of Ref. [12]] is identical to that obtained by Agam and Fishman [9], and thus differs from ours in the limit studied in the present paper, namely, the ballistic regime.

In Fig. 1, we plot the conditional density observed among more than 10^4 levels corresponding to even-even modes obtained in ten different rectangles with a single-point scatterer at the center (see Ref. [3]; also see the Appendix, where a discussion of the method by which the numerical data were generated is given in the general case of many scatterers, as well as a short summary of what systems were studied and what the range of eigenvalues examined). The parameter α was chosen to be zero. This numerical result is compared to the inverse transform of the sum of Eqs. (4) and (5) [13] (dotted line) with $\gamma=2/\pi$ and also to the conditional density for GOE spectra, namely, $1-Y_{2,\mathrm{GOE}}(s)$. The agreement is fair, especially for values of s smaller than unity. For larger values, oscillations are seen in the numerical data which cannot be reproduced by the above ansatz. In Fig. 2, we plot (a) the conditional density observed for levels in a rectangle billiard with three maximum strength scatterers ($\alpha=0$) located at random positions not too near the edge and compare it to our prediction (dotted line) with $\gamma=6/\pi$; and (b) the same for a rectangle billiard with six maximum strength scatterers located at random positions not too near the edge and with $\gamma=12/\pi$. Again, even if numerical data are closer to the GOE behavior, the agreement with our ansatz is still significant.

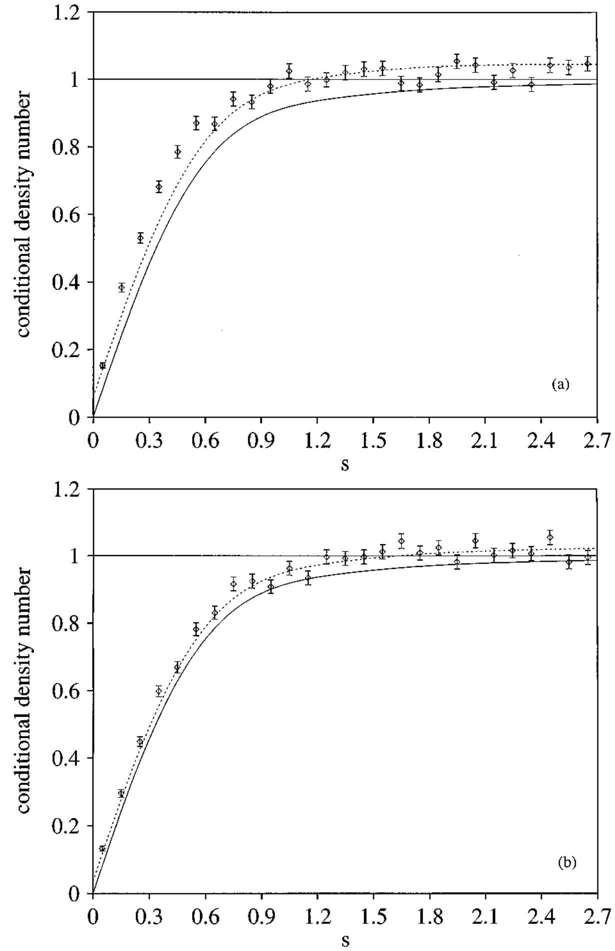

FIG. 2. Same as in Fig. 1 for the cases of a rectangular billiard with (a) three full strength scatterers placed at random, and (b) six full strength scatterers placed at random. The dotted line shows our prediction with (a) $\gamma=6/\pi$ and (b) $\gamma=12/\pi$.

In conclusion, we have proposed a quantitative prediction for the conditional density in rectangle billiards with point scatterers, using a semiclassical analysis of the spectral form factor based on the partition of periodic orbits in two categories: one accounts for the regular behavior of the spectrum correlations at large frequency range, while the other one builds upon scattered orbits which contribute to the ergodic part of the form factor leading to short- and intermediate-range spectral rigidity. This prediction was shown to be in good agreement with numerically observed levels in the rectangular billiard with either a single centered scatterer or a few point scatterers placed at random. It is remarkable that such a simple theory does as well as it does in spite of its limitations.

It is our pleasure to thank D. Sornette for very fruitful discussions on this problem. One of us (O.L.) wishes to acknowledge very stimulating discussions with Eugene Bogomolny.

APPENDIX

We define the bare Green's function, $G^0(\vec{r},\ \vec{s})$ as the harmonic (ω) response at position $\vec{r}$ to a unit source at $\vec{s}$ in a rectangular domain without scatterers. The response $G(\vec{r},\vec{s})$

in the system with j scatterers ($j=1,2,3,\ldots,n$) is a superposition of a response which directly propagates by means of G^0, and several fields, each the bare response to a point source of as yet unknown strengths A_j at positions $\vec{b}_j$. Thus the total field due to a source at $\vec{s}$ is given by

$$G(\vec{r},\vec{s})=G^0(\vec{r},\vec{s})+\sum_j G^0(\vec{r},\vec{b}_j)A_j \tag{A1}$$

for some as-yet-undetermined set of effective source strengths A_j. The quantities A_j in general depend on the source position $\vec{s}$ as well as the positions and scattering strengths of all the scatterers. In the vicinity of scatterer number l, the field is

$$G(\vec{r}\approx\vec{b}_l,\vec{s})=G^0(\vec{b}_l,\vec{s})+\sum_{j\neq l} G^0(\vec{b}_l,\vec{b}_j)A_j + \left[f(\vec{b}_l)-\frac{i}{4}H_0^{(1)}(\omega\|\vec{r}-\vec{b}_l\|)\right]A_l \tag{A2}$$

where we have used the form given in Eq. (16) of Ref. [3] to describe the bare Green's function in the vicinity of its singularity in terms of an incident part (due to multiple reflections from the boundary) and an outgoing part. f is given by [3]

$$f(\vec{b})=\lim_{\vec{r}\to\vec{b}}\left\{G^0(\vec{r},\vec{b})+\frac{i}{4}H_0^{(l)}(\omega\|\vec{r}-\vec{b}\|)\right\}. \tag{A3}$$

The efficient evaluation of f is discussed in Ref. [3]. The ratio of the coefficient of the outgoing part $H_0^{(1)}$ to the incident part is, by definition, the scattering strength of the scatterer (a property of the scatterer and independent of the system in which it is placed):

$$t_l=A_l\Bigg/\left[G^0(\vec{b}_l,\vec{s})+\sum_{j\neq l}G^0(\vec{b}_l,\vec{b}_j)A_j+f(\vec{b}_l)A_l\right]. \tag{A4}$$

The complex scattering strength t_l is related to the real parameter α_l by Eq. (13) of Ref. [3]:

$$t=(i/4+\alpha)^{-1}. \tag{A5}$$

Equation (A4) may be written as an algebraic relation among the unknown source strengths A_j:

$$A_l[t_l f(\vec{b}_l)-1]+t_l\sum_{j\neq l}G^0(\vec{b}_l,\vec{b}_j)A_j+t_l G^0(\vec{b}_l,\vec{s})=0. \tag{A6}$$

Equation (A6) is a linear algebraic relation for the effective source strengths A_j. It is singular, indicating a resonance in the composite system, whenever the matrix of the coefficients of the A_j's has a vanishing determinant. This is the criterion used to find the eigenvalues of the composite system.

The elements of the matrix of coefficients of the A_j's are singular at each of the eigenvalues of the bare systems; thus the matrix can be ill conditioned near these eigenvalues. It may be shown, however, that the determinant has only a simple pole at these eigenvalues. The numerical procedures used to evaluate the determinant were therefore written to take advantage of this feature, and thereby to avoid most of the ill-conditioning.

The conditional densities reported in Fig. 2 were taken from about 22 500 levels for each of the cases (a) and (b). The 22 500 levels were taken from nine sample systems each of about 2500 levels in the range of $50<\omega<80$, consisting of rectangles of size $\pi\times\pi/(\sqrt{5}-1)$. The case of the single scatterer at the center of Fig. 1 was reported in Ref. [3]; the conditional density for this case was based on 10 000 even-even levels in the range $60<\omega<100$.

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[13] A good approximation of the conditional density number for the dressed billiard $g_d(s)$ can be obtained in the limit of large values of γ simply by replacing the product $e^{-\gamma\tau}(1-b_{\rm GOE}(\tau))$ by its leading behavior at small τ, namely, $e^{-\gamma\tau}(\tau/\pi)$, thus yielding the following expression:

$$g_d(s)\approx 1-Y_{2,\rm GOE}(s)-\frac{1}{\pi^2\gamma^2}\frac{1-(s/\gamma)^2}{[1+(s/\gamma)^2]^2}+\frac{1}{2\pi\gamma}\frac{1}{1+(s/2\gamma)^2}.$$

This is presumably valid at large s, but it is found to be a fair approximation for all but the smallest ranges.